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Some Domination Parameters in Generalized Jahangir Graph $J_{n,m}$

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ABSTRACT

Given a graph $G = (V, E)$, a dominating set D is a subset of vertex set V such that any vertex not in D is adjacent to at least one vertex in D. The domination number of a graph G is the minimum size of the dominating sets of G. In this paper, we investigate some domination parameters such as domination number, total domination, independent domination, connected domination, 2-domination, outer-connected domination and doubly connected domination number for a generalized Jahangir graph $J_{n,m}$.

Keywords: Jahangir graph, Domination number, Total domination, Independent domination, 2-domination, Connected domination, Outer-connected domination, Doubly connected domination.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A dominating set is a set D of vertices such that every vertex outside D is dominated by some vertex of D. The domination number of G, denoted by $\gamma(G)$, is the minimum size of dominating set of G. A dominating set D is called an independent dominating set if D is an independent set. The independent domination number of G denoted by $\gamma_i(G)$ is the minimum size of an independent dominating set of G. A dominating set D is a total dominating set of G if every vertex of the graph is adjacent to at least one vertex in D. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum size of a total dominating set of G, see Haynes et al. (1998b).

The subset D of the set of vertices $V(G)$ is a connected dominating set in G if D is a dominating set and the subgraph induced by D is connected. The minimum cardinality of any connected dominating set in G is called the connected domination number of G and it is denoted by $\gamma_c(G)$, see Sampathkumar and Walikar (1979). A 2-dominating set of G is every vertex of $V(G)\setminus D$ has at least two neighbours in D. The 2-domination number of G, denoted by $\gamma_2(G)$ is the minimum cardinality of 2-dominating set of G, see Haynes et al. (1998b). A set D of vertices of a graph G is an outer-connected dominating set if every vertex not in D is adjacent to some vertex in D and the subgraph induced by $V(G) \setminus D$ is connected. The outer-connected domination number $\tilde{\gamma_c}(G)$ is the minimum size of such a set, see Cyman (2007). A set $D \subseteq V(G)$ is a doubly connected dominating set of G if it is dominating and both induced subgraphs D and $V(G) \setminus D$ are connected. The cardinality of a minimum doubly connected dominating set of G is the doubly connected domination number of G and is denoted by $\gamma_{cc}(G)$, see Cyman et al. (2006).

An excellent treatment of the fundamentals of domination can be found in the book by Haynes et. al, see Haynes et al. (1998b). A survey of several advanced topics in domination is given in the book, see Haynes et al. (1998a).

For $n, m \geq 2$, the generalized Jahangir graph $J_{n,m}$ is a graph on $nm + 1$ vertices, i.e., a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}, see Ali et al. (2007). Mojdeh and Ghameshlou (2007) studied some domination parameters for Jahangir graph $J_{2,m}$. Parvathi and Thanga (2012) investigated some domination parameters for Jahangir graph $J_{3,m}$. In this paper, we determine the domination number, total domination, independent domination, connected domination, 2-domination, outer-connected domination and doubly domination number for the generalized Jahangir graph $J_{n,m}$ for $n, m \geq 2$.

Through this paper, we assume that the vertices of $J_{n,m}$ are labeled by v_i, $1 \leq i \leq nm+1$ that v_{nm+1} is adjacent with m vertices $\{v_1, v_{1+n}, v_{1+2n}, ..., v_{1+(m-1)n}\}$ on cycle C_{nm}.

2. The results for the domination parameters of $J_{n,m}$

In this paper, we obtain some results on the domination number, the total domination, independent domination number and 2-domination number of $J_{n,m}$.

Theorem 2.1. *For $n, m \geq 2$,*

$$\gamma(J_{n,m}) = \lceil \frac{nm}{3} \rceil.$$

Proof. It is easy to verify that the set of vertices $S = \cup_{i=0}^{\lceil \frac{nm}{3} \rceil} \{v_{1+3i}\}$ is a domination set of $J_{n,m}$. So, $\gamma(J_{n,m}) \leq \lceil \frac{nm}{3} \rceil$. Let D be a dominating set of $J_{n,m}$. We show that $|D| \geq \lceil \frac{nm}{3} \rceil$. Otherwise, assume $|D| \leq \lceil \frac{nm}{3} \rceil - 1$. We consider following cases:

Case 1: Let $v_{nm+1} \in D$ and $n = 3k$ or $m = 3k$. So, $\frac{nm}{3} - 2$ remained vertices of D is on the cycle C_{nm}. Since, v_{nm+1} is adjacent to m vertices of C_{nm} so, v_{nm+1} dominates them. According to the labeling of vertices $J_{n,m}$, it is clear that v_{nm+1} dominates vertices $\{v_1, v_{1+n}, v_{1+2n}, ..., v_{1+(m-1)n}\}$ of C_{nm}. Therefore, $D \setminus \{v_{nm+1}\}$ must dominate $\frac{2}{3}nm - (m-2)$ vertices of cycle C_{nm}. Since, each vertex on a cycle can dominate two vertices so, $D \setminus \{v_{nm-1}\}$ dominates at most $2(\frac{nm}{3} - 4)$ vertices on C_{nm}. It is a contradiction because there is at least three vertices of $V(J_{n,m})$ that any vertices of D can not dominate them. If $n \equiv 1, 2\, (mod\, 3)$, then similarly there is at least one vertex of $V(J_{n,m})$ that D can not dominate it. It is a contradiction.

Case 2: Let $v_{nm+1} \notin D$, without loss of generality suppose that $v_1 \in D$, then v_1 dominates V_{nm+1}, v_2 and v_m. If $n = 3k$ or $m = 3k$, then $\frac{nm}{3} - 2$ vertices of D must dominate $nm - 3$ vertices on C_{nm}. Since, at least three vertices of $V(J_{n,m})$ are not dominated by vertices of D, we have a contraction. According to similar discussion for $n = 3k + 1$ or $n = 3k + 2$, also, there is a contradiction. □

In Mojdeh and Ghameshlou (2007), the total domination number is obtained for $J_{2,m}$. We study $\gamma_t(J_{3,m})$ and finally $\gamma_t(J_{n,m})$ is obtained for $n \geq 4$ and $m \geq 2$.

Theorem 2.2. *For $m \geq 2$,*

$$\gamma_t(J_{3,m}) = m + 1.$$

Proof. $S = \cup_{i=0}^{m}\{v_{1+3i}\}$ is a total dominating set of $J_{3,m}$. Therefore, $\gamma_t(J_{3,m}) \leq m + 1$. Let D be the total dominating set for $J_{3,m}$ and $|D| \leq m$.
Case 1: Let $v_{3m+1} \in D$. Since, D is a total domination set so, one of vertices $\{v_1, v_4, ..., v_{2m+1}\}$ is in D. Assume $v_1 \in D$. $D \setminus \{v_1, v_{3m+1}\}$ must dominate $2m - 2$ vertices of C_{3m}. But $D \setminus \{v_1, v_{3m+1}\}$ can dominate at most $2m - 4$ vertices. Therefore, it is a contradiction.

Case 2: Let $v_{3m+1} \in D$, we can assume $v_1 \in D$ and since D is the total dominating set, so $v_2 \in D$. $D \setminus \{v_1, v_2\}$ can dominate at most $2m - 4$ vertices on cycle C_{3m}. But the number of remained vertices for dominating $V(C_{nm})$ is $3m - 4$ that it is a contradiction. Therefore $\gamma_t(J_{3,m}) = m + 1$. □

Theorem 2.3. *For $n \geq 4$ and $m \geq 2$,*

$$\gamma_t(J_{n,m}) = \begin{cases} \frac{nm}{2}, & nm \equiv 0 \ (mod\, 4), \\ \frac{nm}{2} + 1, & nm \equiv 2 \ (mod\, 4), \\ \frac{nm+1}{2}, & otherwise. \end{cases}$$

Proof. For a cycle C_n of order n, the total domination number of C_n is as follows (see Amos (2012))

$$\gamma_t(C_n) = \begin{cases} \frac{n}{2}, & n \equiv 0 \text{ (mod 4)}, \\ \frac{n}{2} + 1, & n \equiv 2 \text{ (mod 4)}, \\ \frac{n+1}{2}, & \text{otherwise.} \end{cases}$$

Therefore, if all of vertices of G be on cycle C_{nm} then, this claim is proved. Let D be a total dominating set of $J_{n,m}$ and $v_{nm+1} \in D$. Since, D is the total domination set, we can assume $v_1 \in D$. We consider the path $P = \{v_3, \ldots, v_{nm-1}\}$ of order $nm - 3$. This is sufficient to obtain $\gamma_t(P)$. Thus, $\gamma_t(J_{n,m}) = \gamma_t(P) + 2$. If $nm \equiv 0, 1\, (mod\, 4)$, then $|D|$ is more than $\gamma_t(J_{n,m})$ when $v_{nm+1} \notin D$. If $nm \equiv 2, 3\, (mod\, 4)$, then $|D|$ is equal to $\gamma_t(J_{n,m})$ when $v_{nm+1} \notin D$. So, we can consider $v_{nm+1} \notin D$ and the all of vertices of D are on cycle C_{nm}. □

Theorem 2.4. *For,* $n, m \geq 2$,

$$\gamma_i(J_{n,m}) = \lceil \frac{nm}{3} \rceil.$$

Proof. Since $S = \bigcup_{i=0}^{\lceil \frac{nm}{3} \rceil - 1} \{v_{1+3i}\}$ is an independent dominating set on $J_{n,m}$ then,

$$\gamma_i(J_{n,m}) \leq \lceil \frac{nm}{3} \rceil.$$

On the other hand, using Theorem 2.1 and $\gamma(G) \leq \gamma_i(G)$, we have

$$\lceil \frac{nm}{3} \rceil = \gamma(G) \leq \gamma_i(G) \leq \lceil \frac{nm}{3} \rceil.$$

This completes the proof. □

Theorem 2.5. *For,* $n, m \geq 2$,

$$\gamma_2(J_{n,m}) = \lceil \frac{nm}{2} \rceil.$$

Proof. It is clear that set of vertices $S = \bigcup_{i=0}^{\lceil \frac{nm}{2} \rceil - 1} \{v_{1+2i}\}$ is an 2-connected dominating set of $J_{n,m}$. Therefore

$$\gamma_2(J_{n,m}) \leq |S| = \lceil \frac{nm}{2} \rceil.$$

Let D is a 2-connected dominating set of $J_{n,m}$ and $|D| \leq \lceil \frac{nm}{2} \rceil - 1$. We have the following cases:

Case 1: Let $v_{nm+1} \in D$, then $\lceil \frac{nm}{2} \rceil - 2$ remained vertices of D are on cycle C_{nm}. v_{nm+1} dominates m vertices on cycle C_{nm}. If m or n be even, then $\frac{nm}{2} + 2$ vertices of cycle C_{nm} must dominate by $\frac{nm}{2} - 2$ vertices of D. Thus, there is at least four vertices that can not dominate by all vertices of D that it is a contradiction. Let m and n be odd. v_{nm+1} is adjacent to set of vertices $\{v_1, v_{1+n}, \dots, v_{(m-1)n+1}\}$ and each of $V \setminus D$ must dominate by two vertices of D. So, $\frac{nm-1}{2}$ even vertices on cycle C_{nm} are in D. Therefore, there is at least one odd vertex of $V(C_{nm})$ that any vertex of D can not dominate it. Again we have a contradiction.

Case 2: Let $v_{nm+1} \notin D$. Therefore, $v_1 \in D$. since, D is 2-connected dominating set so, all of odd vertices, i. e., $\{v_3, v_5, \dots, v_{nm}\}$. Therefore, $|D| \geq \lceil \frac{nm}{2} \rceil$. □

Let D be the dominating set of $J_{n,m}$ for $n, m \geq 2$. In next section, we consider some parameters that at least one of sets D or $V \setminus D$ is connected.

3. The results for domination parameters with at least D or $V \setminus D$ is connected

In this section, we study the connected domination, outer-connected domination and doubly connected domination number of $J_{n,m}$ for $n, m \geq 2$.

Lemma 3.1. (*Mojdeh and Ghameshlou (2007)*) $\gamma_c(J_{2,m}) = \lceil \frac{m}{2} \rceil + 1$.

Theorem 3.1. *For, $n \geq 3$ and $m \geq 2$,*

$$\gamma_c(J_{n,m}) = m(n-2) + 1.$$

Proof. Using Lemma 3.1, for $n = 2$ and $m \geq 3$, this result is hold. We prove it for $n, m \geq 3$. The set of vertices $S = \bigcup_{i=0}^{m-1} S_i' \cup \{v_{nm+1}\}$ in which

$$S_i' = \{v_{1+in}, \ldots, v_{(1+i)n-2}\},$$

is a connected dominating set for $J_{n,m}$. Therefore, $\gamma_c(J_{n,m}) \leq m(n-2) + 1$. Let D be a connected dominating set of $J_{n,m}$. We consider the following cases:

Case 1: Let $v_{nm+1} \in D$. Since, D is connected so, one of m adjacent vertices to v_{nm+1} must be in D. Thus, $nm - 3$ vertices are dominated by $D \setminus \{v_1, v_{nm+1}\}$. Since, D is connected so, vertices $\{v_2, \ldots, v_{n-2}\}$ are in D. As the same way, $m(n-2) - 1$ vertices are in $D \setminus \{v_1, v_{nm+1}\}$ and so, $|D| \geq m(n-2) + 1$. It implies that $\gamma_c(J_{n,m}) = m(n-2) + 1$.
Case 2: Let $v_{nm+1} \notin D$ and the vertices of D be on cycle C_{nm}. Since, the connected domination number of C_{nm} is $nm - 2$ so, $|D| \geq |S|$. Therefore, it is a contradiction with being connected domination set D. □

Theorem 3.2. *For $m \geq 2$,*

$$\tilde{\gamma_c}(J_{2,m}) = \begin{cases} \frac{2m}{3}, & m \equiv 0 \ (mod\ 3) \\ \lfloor \frac{2m}{3} \rfloor + 1, & m \equiv 1, 2 \ (mod\ 3) \end{cases}$$

Proof. (a) Let $m = 3k$, then S as following is an outer-connected dominating set for $J_{2,m}$.

$$S = \bigcup_{i=1}^{\frac{2m}{3}} \{v_{3i}\}.$$

So, $\tilde{\gamma_c}(J_{2,m}) \leq \frac{2m}{3}$. Let D be an outer-connected dominating set of $J_{2,m}$ and $|D| \leq \frac{2m}{3} - 1$.

Case 1: If $v_{2m+1} \in D$, then v_{2m+1} dominates m vertices on $J_{2,m}$. Since, D is the outer-connected domination and $V \setminus D$ is connected so, $D \setminus \{v_{2m+1}\}$ must dominate vertices on C_{2m}. The outer-connected domination number of cycle C_{2m} is $2m-2$ (see Cyman (2007)). It is a contradiction.
Case 2: If $v_{2m+1} \notin D$, then we can assume $v_1 \in D$. So, $\frac{2m}{3} - 2$ remained vertices of D must dominate $2m-3$ vertices of C_{2m}. It is a contradiction. Because at least vertices of $V(C_{2m})$ are not dominated by any vertex of D.

(b) Now, we assume $m = 3k+1$ or $m = 3k+2$. We consider the following set as an outer-connected dominating set of $J_{2,m}$:

$$S = \bigcup_{i=1}^{\lfloor \frac{2m}{3} \rfloor} \{v_{3i}\} \cup \{v_{2m}\}.$$

Therefore, $\tilde{\gamma}_c(J_{2,m}) \leq \lfloor \frac{2m}{3} \rfloor + 1$. Let D be an outer-connected dominating set of $J_{2,m}$ and $|D| \leq \lfloor \frac{2m}{3} \rfloor$. Using similar discussion in (a), we can assume $v_1 \in D$. So, $\lfloor \frac{2m}{3} \rfloor - 1$ vertices of D are on cycle C_{2m} and they dominate $2m-3$ vertices of C_{2m}. It is a contradiction since, at least one vertex of C_{2m} is not dominated by D. □

Theorem 3.3. *For $m \geq 2$ and $n \geq 3$, $\tilde{\gamma}_c(J_{n,m}) = m(n-2)+2$.*

Proof. We consider $D = \bigcup_{i=1}^{m-1} D_i$ where $D_1 = \{v_1, v_2, \ldots, v_n\}$ and $D_i = \{v_{in+3}, \ldots, v_{(i+1)n}\}$ for $1 \leq i \leq m-1$. D is a connected dominating set of $J_{n,m}$. Therefore, $\tilde{\gamma}_c(J_{n,m}) \leq m(n-2)+2$. On the other hand, since $\tilde{\gamma}_c(C_n) = n-2$ for cycle C_n of order n (see Cyman (2007)) and using similar discussion in Theorem 2.5, we can easy verify $\tilde{\gamma}_c(J_{n,m}) = m(n-2)+2$. □

Theorem 3.4. *For, $m, n \geq 2$, $\gamma_{cc}(J_{n,m}) = nm-2$.*

Proof. Let D be a doubly connected dominating set of $J_{n,m}$. Since, D and $V \setminus D$ are connected sets so, we must select all of vertex of D on C_{nm}. For cycle C_n of order n, $\gamma_{cc}(J_{n,m}) = nm-2$ (see Cyman et al. (2006)). Therefore, $\gamma_{cc}(J_{n,m}) = nm-2$. □

4. Conclusions

In this paper, we studied domination parameters such as domination number, total domination, independent domination, connected domination, 2-domination, outer-connected domination and doubly connected domination number for a generalized Jahangir graph $J_{n,m}$.

For many graph parameters, criticality is very important concept. Much has been written about graphs for which a parameter (for example domination number) increases or decreases whenever an edge or vertex is removed or added. Brigham et al. Brigham et al. (1998) began the study of graphs where domination number decreases on the removal of any vertex. This problem is then called as the domination critical of graphs. Thus, we first pose the following interesting problem.

Problem 1 Characterize the domination critical of Jahangir graph $J_{n,m}$ for $n, m \geq 2$.

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References

Ali, K., Bakoro, E. T., and Tomescu, I. (2007). On the Ramsey number of paths and Jahangir graph $J_{3,m}$. *The 3rd International Conference on 21st Century Mathematics, GC University Lahore, Pakistan.*

Amos, D. (2012). On total domination in graphs. *Senior Project, University of Houston.*

Brigham, R. C., Chin, P. Z., and Dutton, R. D. (1998). Vertex domination-critical graphs. *Networks*, 18:173–179.

Cyman, J. (2007). The outer-connected domination number of a graph. *Australas J. Comb.*, 38:35–46.

Cyman, J., Lemańska, M., and Raczek, J. (2006). On the doubly connected domination number of a graph. *Central European Journal of Mathematics*, 4:34–45.

Haynes, T. W., Hedetniemi, S. T., and Slater, P. J. (1998a). *Domination in Graphs - Advanced Topics.* Marcel Dekker Inc., New York.

Haynes, T. W., Hedetniemi, S. T., and Slater, P. J. (1998b). *Fundamentals of Domination in Graphs*. Marcel Dekker Inc., New York.

Mojdeh, D. A. and Ghameshlou, A. N. (2007). Domination in Jahangir graph $J_{2,m}$. *International Journal of Contemporary Mathematical Sciences*, 2(24):1193–1199.

Parvathi, N. and Thanga, R. D. (2012). Study of domination in Jahangir graph $J_{3,m}$. *National Conference on Mathematical Techniques and Its Applications (NCMTA), Chennai, SRM University*, pages 234–240.

Sampathkumar, E. and Walikar, H. (1979). The connected domination number of a graph. *Journal of Mathematical and Physical Sciences*, 13:607–613.